

Hypercyclic operators on topological vector spaces

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Abstract

Bonet, Frerick, Peris and Wengenroth constructed a hypercyclic operator on the locally convex direct sum of countably many copies of the Banach space ℓ_1 . We extend this result. In particular, we show that there is a hypercyclic operator on the locally convex direct sum of a sequence $\{X_n\}_{n \in \mathbb{N}}$ of Fréchet spaces if and only if each X_n is separable and there are infinitely many $n \in \mathbb{N}$ for which X_n is infinite dimensional. Moreover, we characterize inductive limits of sequences of separable Banach spaces which support a hypercyclic operator.

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1 Introduction

Unless stated otherwise, all vector spaces in this article are over the field \mathbb{K} , being either the field \mathbb{C} of complex numbers or the field \mathbb{R} of real numbers and all topological spaces *are assumed to be Hausdorff*. When we speak of a topological vector space X , we mean a *non-trivial* topological vector space: $X \neq \{0\}$. As usual, \mathbb{Z} is the set of integers, \mathbb{Z}_+ is the set of non-negative integers and \mathbb{N} is the set of positive integers. Symbol $L(X, Y)$ stands for the space of continuous linear operators from a topological vector space X to a topological vector space Y . We write $L(X)$ instead of $L(X, X)$ and X' instead of $L(X, \mathbb{K})$. By a *quotient* of a topological vector space X we mean the space X/Y , where Y is a closed linear subspace of X . For a subset A of a vector space X , $\text{span}(A)$ stands for the linear span of A . For brevity, we say *locally convex space* for a locally convex topological vector space. We also often say '*operator*' instead of '*continuous linear operator*'. Recall also that if X is a topological vector space, then $A \subset X'$ is called *uniformly equicontinuous* if there exists a neighborhood U of zero in X such that $|f(x)| \leq 1$ for any $x \in U$ and $f \in A$. A subset B of a topological vector space X is called *bounded* if for any neighborhood U of zero in X , a scalar multiple of U contains B . A subset A of a vector space is called *balanced* if $zx \in A$ whenever $x \in A$, $z \in \mathbb{K}$ and $|z| \leq 1$. A subset D of a topological vector space X is called a *disk* if D is convex, balanced and bounded. For a disk D , the space $X_D = \text{span}(D)$ is endowed with the norm p_D , being the Minkowskii functional [16] of the set D . Boundedness of D implies that the norm topology of X_D is stronger than the topology inherited from X . If the normed space X_D is complete, then D is called a *Banach disk*. It is well-known [4] that a compact disk is a Banach disk.

Definition 1.1. We say that a sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ of elements of a locally convex space X is an ℓ_1 -sequence if $x_n \rightarrow 0$ and the series $\sum_{n=0}^{\infty} a_n x_n$ converges in X for each $a \in \ell_1$.

The following lemma is a standard and elementary fact, see for instance [4]. See also [15] for its version for general topological vector spaces.

Lemma 1.2. *Let $\{x_n\}_{n \in \mathbb{Z}_+}$ be an ℓ_1 -sequence in a locally convex space X . Then the closed balanced convex hull D of $\{x_n\}_{n \in \mathbb{Z}_+}$ is a compact and metrizable disk. Moreover, the Banach space X_D is separable and $\text{span}\{x_n : n \in \mathbb{Z}_+\}$ is dense in X_D .*

We say that τ is a *locally convex topology* on a vector space X if (X, τ) is a locally convex space. We say that the topology τ of a topological vector space X is *weak* if τ is exactly the weakest topology making each $f \in Y$ continuous for some linear space Y of linear functionals on X separating points of X . An \mathcal{F} -space is a complete metrizable topological vector space. A locally convex \mathcal{F} -space is called a *Fréchet space*. If $\{X_\alpha : \alpha \in A\}$ is a family of locally convex spaces, then their (locally convex) *direct sum* is the algebraic direct sum $X = \bigoplus_{\alpha \in A} X_\alpha$ of the vector spaces X_α endowed with the strongest locally convex topology, which induces the original topology on each X_α . Let $\{X_n\}_{n \in \mathbb{Z}_+}$ be a sequence of vector spaces such that each X_n is a subspace of X_{n+1} and each X_n carries its own locally convex topology τ_n such that τ_n is (maybe non-strictly) stronger than $\tau_{n+1}|_{X_n}$. Then the (locally convex) *inductive limit* of the sequence $\{X_n\}$ is the space $X = \bigcup_{n=0}^{\infty} X_n$ endowed with the strongest locally convex topology τ such that $\tau|_{X_n} \subseteq \tau_n$ for each $n \in \mathbb{Z}_+$. In other words, a convex set U is a neighborhood of zero in X if and only if $U \cap X_n$ is a neighborhood of zero in X_n for each $n \in \mathbb{Z}_+$. An *LB-space* is an inductive limit of a sequence of Banach spaces. An *LB_S-space* is an inductive limit of a sequence of separable Banach spaces. We use symbol φ to denote the locally convex direct sum of countably many copies of the one-dimensional space \mathbb{K} , while ω is the product of countably many copies of \mathbb{K} .

Let X and Y be topological spaces and \mathcal{F} be a family of continuous maps from X to Y . An element $x \in X$ is called *universal* for \mathcal{F} if the orbit $\{Tx : T \in \mathcal{F}\}$ is dense in Y and \mathcal{F} is said to be *universal* if it has a universal element. We say that $\{T_n : n \in \mathbb{Z}_+\}$ is *hereditarily universal* if $\{T_n : n \in A\}$ is universal for each infinite subset A of \mathbb{Z}_+ . If $T : X \rightarrow X$ is continuous, then T is called *transitive* if for any non-empty open subsets U and V of X , there is $n \in \mathbb{Z}_+$ such that $T^n(U) \cap V \neq \emptyset$. The map T is said to be *mixing* if for any non-empty open subsets U and V of X , there is $n \in \mathbb{Z}_+$ such that $T^k(U) \cap V \neq \emptyset$ for each $k \geq n$. Recall that a topological space X is called a *Baire space* if the intersection of countably many dense open subsets of X is dense in X . According to the classical Baire theorem, complete metric spaces are Baire.

Let X be a topological vector space and $T \in L(X)$. A vector $x \in X$ is called a *cyclic vector* for T if $\text{span}\{T^n x : n \in \mathbb{Z}_+\}$ is dense in X and T is called *cyclic* if it has a cyclic vector. Recall also that T is called *multicyclic* if $\text{span}\{T^n x : n \in \mathbb{Z}_+, x \in A\}$ is dense in X for some finite set $A \subset X$. T is called *hypercyclic* if $\{T^n : n \in \mathbb{Z}_+\}$ is universal and a universal element for this family is called a *hypercyclic vector* for T . Similarly, T is *supercyclic* if $\{zT^n : n \in \mathbb{Z}_+, z \in \mathbb{K}\}$ is universal and a universal element for this family is a *supercyclic vector* for T . Finally, T is called *hereditarily hypercyclic* if $\{T^n : n \in \mathbb{Z}_+\}$ is hereditarily universal. Study of linear operators from above classes goes under the name 'chaotic linear dynamics'. We refer to the book [2] and references therein. The following proposition is a collection of well-known observations, see, for instance, [2].

Proposition 1.3. *Any hypercyclic operator is transitive and any hereditarily hypercyclic operator is mixing. If the underlying space is Baire separable and metrizable, then the converse implications hold: any transitive operator is hypercyclic and any mixing operator is hereditarily hypercyclic.*

The question of existence of hypercyclic operators on various types of topological vector spaces was intensely studied. Ansari [1] and Bernal-González [3], answering a question of Herrero, showed independently that any separable infinite dimensional Banach space supports a hypercyclic operator. Using the same idea, Bonnet and Peris [5] proved that there is a hypercyclic operator on any separable infinite dimensional Fréchet space and demonstrated that there is a hypercyclic operator on an inductive limit X of a sequence X_n for $n \in \mathbb{Z}_+$ of separable Banach spaces provided there is $n \in \mathbb{Z}_+$ for which X_n is dense in X . Grivaux [7] observed that hypercyclic operators in [1, 3, 5] are mixing and therefore hereditarily hypercyclic. The following proposition is a version of a theorem in [1, 5] and can be used to formalize this observation. We provide a sketch of a proof for sake of completeness.

Proposition 1.4. *Let X be a locally convex space, $\{x_n\}_{n \in \mathbb{Z}_+}$ be an ℓ_1 -sequence in X with dense linear span and $\{f_n\}_{n \in \mathbb{Z}_+}$ be a uniformly equicontinuous sequence in X' such that $f_n(x_m) = 0$ whenever*

$n \neq m$ and $f_n(x_n) \neq 0$ for each $n \in \mathbb{Z}_+$. Then the formula $Tx = \sum_{n=0}^{\infty} 2^{-n} f_{n+1}(x) x_n$ defines a continuous linear operator on X such that $I + T$ is hereditarily hypercyclic.

Proof. By Lemma 1.2, the closed balanced convex hull D of $\{x_n : n \in \mathbb{Z}_+\}$ is a compact disk in X , X_D is a separable Banach space and $E = \text{span}\{x_n : n \in \mathbb{Z}_+\}$ is dense in X_D . Observe that the restriction T_D of T to X_D belongs to $L(X_D)$. By Theorem 2.2 in [2], if Y is a topological vector space and $S \in L(Y)$ is such that the linear span $\Lambda(S)$ of the union of $\ker S^n \cap S^n(Y)$ for $n \in \mathbb{N}$ is dense in Y , then $I + S$ is mixing. Since $E \subseteq \Lambda(T_D)$, we can apply this result to $S = T_D$, to see that $I + T_D$ is a mixing operator on X_K . By Proposition 1.3, $I + T_D$ is hereditarily hypercyclic. Since X_D is dense in X and carries a topology stronger than the one inherited from X , $I + T$ is also hereditarily hypercyclic. \square

Applying an inductive construction exactly as in the proof in [5] of existence of hypercyclic operators on separable infinite dimensional Fréchet spaces, one can verify the following proposition, see also [15] for a detailed proof in a more general setting.

Proposition 1.5. *Let X be a locally convex space such that there exist an ℓ_1 -sequence in X with dense linear span and a linearly independent uniformly equicontinuous sequence in X' . Then there are an ℓ_1 -sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in X with dense linear span and a uniformly equicontinuous sequence $\{f_n\}_{n \in \mathbb{Z}_+}$ in X' such that $f_n(x_m) = 0$ whenever $n \neq m$ and $f_n(x_n) \neq 0$ for each $n \in \mathbb{Z}_+$.*

Let X be a locally convex space. It is easy to see that the following conditions are equivalent: (a) the topology of X is not weak, (b) there is a linearly independent uniformly equicontinuous sequence $\{f_n\}_{n \in \mathbb{Z}_+}$ in X' and (c) there is a continuous seminorm p on X such that $\ker p$ has infinite codimension in X . Thus, combining Propositions 1.4 and 1.5, we arrive to the following result containing all the mentioned existence theorems, except for the space ω , which is a separate case in [5] anyway.

Theorem 1.6. *Let X be a locally convex space such that the topology of X is not weak and there is an ℓ_1 -sequence in X with dense linear span. Then there exists a hereditarily hypercyclic $T \in L(X)$.*

1.1 Results

The simplest space with no ℓ_1 -sequences with dense linear span is φ . Bonet and Peris [5] observed that there are no supercyclic operators on φ . On the other hand, Bonet, Frerick, Peris and Wengenroth [6] constructed a hypercyclic operator on the locally convex direct sum X of countably many copies of ℓ_1 . Clearly, X is a complete LB_S -space with no ℓ_1 -sequences with dense linear span. We characterize the LB_S -spaces, supporting a hypercyclic operator.

Theorem 1.7. *Let X be the inductive limit of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of separable Banach spaces. Then the following conditions are equivalent:*

- (1.7.1) X supports no hypercyclic operators;
- (1.7.2) X supports no cyclic operators with dense range;
- (1.7.3) X is isomorphic to $Y \times \varphi$, where Y is the inductive limit of a sequence $\{Y_n\}_{n \in \mathbb{Z}_+}$ of separable Banach spaces such that Y_0 is dense in Y ;
- (1.7.4) for any sufficiently large n , $\overline{X_{n+1}}/\overline{X_n}$ is finite dimensional and the set $\{n \in \mathbb{Z}_+ : \overline{X_{n+1}} \neq \overline{X_n}\}$ is infinite, where $\overline{X_k}$ is the closure of X_k in X .

The proof is based upon the following result, which is of independent interest.

Theorem 1.8. *Let X be a topological vector space, which has no quotients isomorphic to φ . Then there are no cyclic operators with dense range on $X \times \varphi$.*

The following two theorems provide more generalizations of the main result of [6].

Theorem 1.9. *There is a hypercyclic operator on the direct sum of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of separable Fréchet spaces if and only if X_n are infinite dimensional for infinitely many n .*

Theorem 1.10. *Let X be the direct sum of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of locally convex spaces such that the topology of each X_n is not weak and each X_n admits an ℓ_1 -sequence with dense linear span. Then there is a hypercyclic operator on X .*

2 ℓ_1 -sequences and equicontinuous sets

Lemma 2.1. *Let Y_0 and Y_1 be closed linear subspaces of a locally convex space Y such that $Y_0 \subset Y_1$ and the topology of Y_1/Y_0 is not weak. Then there is a uniformly equicontinuous sequence $\{f_n : n \in \mathbb{Z}_+\}$ in Y' such that $\varphi \subseteq \{\{f_n(y)\}_{n \in \mathbb{Z}_+} : y \in Y_1\}$ and $f_n|_{Y_0} = 0$ for each $n \in \mathbb{Z}_+$.*

Proof. Since the topology of Y_1/Y_0 is not weak, there is a continuous seminorm \tilde{p} on Y_1/Y_0 such that $\ker \tilde{p}$ has infinite codimension in Y_1/Y_0 . Clearly the seminorm p on Y_1 defined by the formula $p(y) = \tilde{p}(y + Y_0)$ is continuous, $\ker p$ has infinite codimension and $Y_0 \subseteq \ker p$. In particular $Y_p = Y_1/\ker p$ endowed with the norm $\|x + \ker p\| = p(x)$ is an infinite dimensional normed space. Since every infinite dimensional normed space admits a biorthogonal sequence, we can choose sequences $\{y_n\}_{n \in \mathbb{Z}_+}$ in Y_1 and $\{g_n\}_{n \in \mathbb{Z}_+}$ in Y'_p such that $\|g_n\| \leq 1$ and $g_n(y_k + \ker p) = \delta_{n,k}$ for any $n, k \in \mathbb{Z}_+$, where $\delta_{n,k}$ is the Kronecker delta. Now let $h_n : Y_1 \rightarrow \mathbb{K}$, $h_n(y) = g_n(y + \ker p)$. The properties of g_n can be rewritten in terms of h_n in the following way: $|h_n(y)| \leq p(y)$ and $h_n(y_k) = \delta_{n,k}$ for any $n, k \in \mathbb{Z}_+$ and $y \in Y_1$. Since any continuous seminorm on a subspace of a locally convex space extends to a continuous seminorm on the entire space [12, 4], there is a continuous seminorm q on Y such that $q|_{Y_1} = p$. By Hahn–Banach theorem, we can find $f_n \in Y'$ such that $f_n|_{Y_1} = h_n$ and $|f_n(y)| \leq q(y)$ for any $n \in \mathbb{Z}_+$ and $y \in Y$. Since $y_k \in Y_1$, the equalities $f_n(y_k) = h_n(y_k) = \delta_{n,k}$ imply that $\varphi \subseteq \{\{f_n(y)\}_{n \in \mathbb{Z}_+} : y \in Y_1\}$. The inequalities $|f_n(y)| \leq q(y)$ guarantee uniform equicontinuity of $\{f_n : n \in \mathbb{Z}_+\}$. The same inequalities and the inclusion $Y_0 \subseteq \ker p \subseteq \ker q$ ensure that $f_n|_{Y_0} = 0$ for each $n \in \mathbb{Z}_+$. \square

Applying Lemma 2.1 in the case $Y = Y_1$ and $Y_0 = \{0\}$, we get the following corollary.

Corollary 2.2. *Let Y be a locally convex space, whose topology is not weak. Then there exists a uniformly equicontinuous sequence $\{f_n : n \in \mathbb{Z}_+\}$ in Y' such that $\varphi \subseteq \{\{f_n(y)\}_{n \in \mathbb{Z}_+} : y \in Y\}$.*

Lemma 2.3. *Let X and Y be locally convex spaces such that the topology of X is not weak and Y admits an ℓ_1 -sequence $\{y_n\}_{n \in \mathbb{Z}_+}$ with dense linear span. Then there is $T \in L(X, Y)$ such that $T(X)$ is dense in Y .*

Proof. By Corollary 2.2, there is a uniformly equicontinuous sequence $\{f_n\}_{n \in \mathbb{Z}_+}$ in X' such that $\varphi \subseteq \{\{f_n(x)\}_{n \in \mathbb{Z}_+} : x \in X\}$. Consider $T : X \rightarrow Y$ defined by the formula $Tx = \sum_{n=0}^{\infty} 2^{-n} f_n(x) y_n$, where the series converges since $\{f_n(x)\}$ is bounded and $\{y_n\}$ is an ℓ_1 -sequence. Since $\{f_n\}$ is uniformly equicontinuous, there is a continuous seminorm p on X such that $|f_n(x)| \leq p(x)$ for any $x \in X$ and $n \in \mathbb{Z}_+$. Since $\{y_n\}$ is an ℓ_1 -sequence, Lemma 1.2 implies that the closed convex balanced hull D of $\{y_n : n \in \mathbb{Z}_+\}$ is a disk. Since $q(y_n) \leq 1$, it is easy to see that $q(Tx) \leq 2p(x)$ for each $x \in X$, where q is the Minkowskii functional of D . Hence $T \in L(X, Y_D)$. Since the topology of Y_D is stronger than the one inherited from Y , $T \in L(X, Y)$. The inclusion $\varphi \subseteq \{\{f_n(x)\}_{n \in \mathbb{Z}_+} : x \in X\}$ implies that $T(X)$ contains $\text{span}\{y_n : n \in \mathbb{Z}_+\}$, which is dense in Y . Thus T has dense range. \square

Remark 2.4. Obviously if X and Y are topological vector spaces and the topology of X is not weak, then the topology of $X \times Y$ is not weak. It is also easy to see that the class of locally convex spaces admitting an ℓ_1 -sequence with dense linear span contains separable Fréchet spaces and is closed under finite or countable products.

3 Operators on $\varphi \times X$

Recall [12] that φ can be interpreted as a linear space of countable algebraic dimension carrying the strongest locally convex topology. In this section we discuss certain properties of φ , mainly those related to continuous linear operators. It is well known [12] that φ is complete and all linear subspaces of φ are closed. Moreover, infinite dimensional subspaces of φ are isomorphic to φ . It is also well-known that for any topology θ on φ such that (φ, θ) is a topological vector space, θ is weaker than the original topology of φ . This observation implies the following lemma.

Lemma 3.1. *For any topological vector space X , any linear map $T : \varphi \rightarrow X$ is continuous.*

Lemma 3.2. *Let X be a topological vector space and $T : X \rightarrow \varphi$ be a surjective continuous linear operator. Then X is isomorphic to $\varphi \times \ker T$.*

Proof. Since T is linear and surjective, there exists a linear map $S : \varphi \rightarrow X$ such that $TS = I$. By Lemma 3.1, S is continuous. Consider the linear maps $A : \varphi \times \ker T \rightarrow X$ and $B : X \rightarrow \varphi \times \ker T$ defined by the formulae $A(u, y) = y + Su$ and $Bx = (Tx, x - STx)$ respectively. It is easy to verify that A and B are continuous, $AB = I$ and $BA = I$. Hence B is a required isomorphism. \square

Corollary 3.3. *Let X be a topological vector space. Then the following are equivalent:*

- (3.3.1) X is isomorphic to a space of the shape $Y \times \varphi$, where Y is a topological vector space;
- (3.3.2) X has a quotient isomorphic to φ ;
- (3.3.3) there is $T \in L(X, \varphi)$ such that $T(X)$ is infinite dimensional.

Proof. The implications (3.3.1) \implies (3.3.2) \implies (3.3.3) are trivial. Assume that there is $T \in L(X, \varphi)$ with infinite dimensional $T(X)$. Since any infinite dimensional linear subspace of φ is isomorphic to φ , $T(X)$ is isomorphic to φ . Hence there is a surjective $S \in L(X, \varphi)$. By Lemma 3.2, X is isomorphic to $Y \times \varphi$, where $Y = \ker S$. Hence (3.3.3) implies (3.3.1). \square

3.1 Multicyclic operators on φ

Clearly φ is isomorphic to the space \mathcal{P} of all polynomials over \mathbb{K} endowed with the strongest locally convex topology. The shift operator on φ is similar to the operator

$$M : \mathcal{P} \rightarrow \mathcal{P}, \quad Mp(z) = zp(z). \quad (3.1)$$

Lemma 3.4. *An operator $T \in L(\varphi)$ is cyclic if and only if T is similar to M .*

Proof. Since 1 is a cyclic vector for M , any operator similar to M is cyclic. Now let $T \in L(\varphi)$ and $x \in \varphi$ be a cyclic vector for T . Then $T^n x$ for $n \in \mathbb{Z}_+$ are linearly independent. Indeed, otherwise their span is finite dimensional, which contradicts cyclicity of x for T . Since any linear subspace of φ is closed, $\{T^n x : n \in \mathbb{Z}_+\}$ is an algebraic basis of φ . Then the linear map $J : \mathcal{P} \rightarrow \varphi$, $Jp = p(T)x$ is invertible. By Lemma 3.1, J and J^{-1} are continuous. It is easy to see that $T = JMJ^{-1}$. Hence T is similar to M . \square

Lemma 3.5. *Let $T \in L(\varphi)$. Then the following conditions are equivalent:*

- (3.5.1) T is multicyclic;
- (3.5.2) there exist $k \in \mathbb{N}$ and a linear subspace Y of φ of finite codimension such that $T(Y) \subseteq Y$ and the restriction $T|_Y \in L(Y)$ is similar to M^k , where M is defined in (3.1).

Proof. First, assume that (3.5.2) is satisfied. Pick a finite dimensional subspace Z of φ such that $\varphi = Z \oplus Y$. Since $T|_Y$ is similar to M^k , there is an invertible linear operator $J : \mathcal{P} \rightarrow Y$ for which $T|_Y = JM^k J^{-1}$. Let $L = Z + J(\mathcal{P}_k)$, where $\mathcal{P}_n = \{p \in \mathcal{P} : \deg p < n\}$. Clearly L is finite dimensional. By the equality $T|_Y = JM^k J^{-1}$, $L + T(L) + \dots + T^{n-1}(L) \supseteq Z + J(\mathcal{P}_{nk})$ for any $n \in \mathbb{N}$. Hence

the linear span of the union of $T^j(L)$ for $j \in \mathbb{Z}_+$ contains $Z + J(\mathcal{P}) = Z + Y = \varphi$. Since L is finite dimensional, T is multicyclic. That is, (3.5.2) implies (3.5.1).

Assume that T is multicyclic. Then there is a subspace L of φ such that

$$\dim L = n \in \mathbb{N} \quad \text{and} \quad \text{span}\{T^k x : x \in L, k \in \mathbb{Z}_+\} = \varphi, \quad (3.2)$$

where we again use the fact that any linear subspace of φ is closed and therefore any dense subspace of φ coincides with φ . We say that $x_1, \dots, x_m \in \varphi$ are T -independent if for any polynomials p_1, \dots, p_m , the equality $p_1(T)x_1 + \dots + p_m(T)x_m = 0$ implies $p_1 = \dots = p_m = 0$. Otherwise, x_1, \dots, x_m are T -dependent. Since T -independence implies linear independence, any T -independent subset of L has at most n elements. Let A be a T -independent subset of L of maximal cardinality $k \leq n$. Since A is linearly independent, there is $B \subset L$ of cardinality $n - k$ such that $A \cup B$ is a basis in L . By definition of k , for any $b \in B$, $A \cup \{b\}$ is T -dependent and therefore, using T -independence of A , we can find polynomials p_b and $p_{b,a}$ for $a \in A$ such that $p_b \neq 0$ and $p_b(T)b = \sum_{a \in A} p_{b,a}(T)a$. Let $m = 0$ if $B = \emptyset$ and $m = \max\{\deg p_b : b \in B\}$ otherwise. Consider the spaces

$$Z = \text{span}\{T^j b : b \in B, 0 \leq j \leq m\} \quad \text{and} \quad Y = \text{span}\{T^j a : a \in A, j \in \mathbb{Z}_+\}.$$

Then Z is finite dimensional and $T(Y) \subseteq Y$. Obviously,

$$T^j a \in Y \subseteq Y + Z \quad \text{for any } a \in A \text{ and } j \in \mathbb{Z}_+. \quad (3.3)$$

Let $b \in B$ and $j \in \mathbb{Z}_+$. Since $p_b \neq 0$ and $\deg p_b \leq m$, we can find polynomials q, r such that $\deg r < m$ and $t^j = q(t)p_b(t) + r(t)$. Then $T^j b = q(T)p_b(T)b + r(T)b$ and $r(T)b \in Z$ since $\deg r < m$. Next, $p_b(T)b \in Y$ since $p_b(T)b = \sum_{a \in A} p_{b,a}(T)a$ and $q(T)p_b(T)b \in Y$ since $T(Y) \subseteq Y$. Thus

$$T^j b \in Y + Z \quad \text{for any } b \in B \text{ and } j \in \mathbb{Z}_+. \quad (3.4)$$

Since $A \cup B$ is a basis of L , (3.3) and (3.4) imply that $T^j(L) \subseteq Y + Z$ for each $j \in \mathbb{Z}_+$. By (3.2), $\varphi = Y + Z$. Since Z is finite dimensional, Y has finite codimension in φ . In particular, Y is non-trivial and $A \neq \emptyset$. That is, $1 \leq k \leq n$ and $A = \{a_1, \dots, a_k\}$. Now consider the linear operator $J : \mathcal{P} \rightarrow Y$, which sends the monomial t^l to $T^j a_s$, where $j \in \mathbb{Z}_+$ and $s \in \{1, \dots, k\}$ are uniquely defined by $l + 1 = jk + s$. By definition of Y , J is onto. By T -independence of A , J is one-to-one. By definition of J , $Jt^{l+k} = TJt^l$. Hence J is invertible and $JM^k = T|_Y J$. That is, M^k and $T|_Y$ are similar. Thus (3.5.1) implies (3.5.2). \square

Corollary 3.6. *Let T be a multicyclic operator on φ . Then T is not onto.*

Proof. By Lemma 3.5, $\varphi = Y \oplus Z$, where Z has finite dimension $m \in \mathbb{Z}_+$, $T(Y) \subseteq Y$ and $T|_Y$ is similar to M^k for some $k \in \mathbb{N}$. Since $T|_Y$ is similar to M^k , $T^{m+1}(Y)$ has codimension $k(m+1) > m$ in Y . Hence $\dim \varphi / T^{m+1}(Y) > m$. On the other hand, $\dim T^{m+1}(Z) \leq \dim Z = m$. Thus $T^{m+1}(\varphi) = T^{m+1}(Z) + T^{m+1}(Y)$ has positive codimension in φ . Hence T^{m+1} is not onto and so is T . \square

3.2 Proof of Theorem 1.8

Let X be a topological vector space with no quotients isomorphic to φ . We have to show that there are no cyclic operators with dense range on $X \times \varphi$. Assume the contrary and let $T \in L(X \times \varphi)$ be a cyclic operator with dense range. Clearly $T(x, u) = (Ax + Bu, Cx + Du)$ for any $(x, u) \in X \times \varphi$, where $A \in L(X)$, $B \in L(\varphi, X)$, $C \in L(X, \varphi)$ and $D \in L(\varphi)$. Since T is cyclic, we can pick a vector $(x, u) \in X \times \varphi$ such that $E = \text{span}\{T^k(x, u) : k \in \mathbb{Z}_+\}$ is dense in $X \times \varphi$. Since T has dense range, T^m has dense range for any $m \in \mathbb{Z}_+$. Thus $E_m = T^m(E) = \text{span}\{T^k(x, u) : k \geq m\}$ is dense in $X \times \varphi$ for any $m \in \mathbb{Z}_+$. Let $T^k(x, u) = (x_k, u_k)$, where $x_k \in X$ and $u_k \in \varphi$. Since $E_m = \text{span}\{(x_k, u_k) : k \geq m\}$ are dense in $X \times \varphi$, $F_m = \text{span}\{u_k : k \geq m\}$ are dense in φ . Hence $F_m = \varphi$ for any $m \in \mathbb{Z}_+$. Since X has no quotients isomorphic to φ , Lemma 3.3 implies that $L = \text{span}(C(X) \cup \{u\})$ is a finite

dimensional subspace of φ . Clearly $u_0 = u \in L$ and $u_{k+1} = Cx_k + Du_k \in Du_k + L$ for any $k \in \mathbb{Z}_+$. It follows that each u_k belongs to the space spanned by the union of $D^m(L)$ for $m \in \mathbb{Z}_+$. Since L is finite dimensional and the linear span of all u_k is φ , D is multicyclic. By Lemma 3.5, we can decompose φ into a direct sum $\varphi = Y \oplus Z$, where Z is finite dimensional, $D(Y) \subseteq Y$ and $D|_Y$ is similar to M^n for some $n \in \mathbb{N}$. Then there is an invertible $J \in L(Y, \mathcal{P})$ such that $D|_Y = J^{-1}M^nJ$. Let also $P \in L(\varphi)$ be the linear projection onto Y along Z . We consider two cases.

Case 1. The sequence $\{\deg JPu_k\}$ is bounded. In this case $\text{span}\{JPu_k : k \in \mathbb{Z}_+\}$ is finite dimensional. Since JP has finite dimensional kernel, $F_0 = \text{span}\{u_k : k \in \mathbb{Z}_+\}$ is finite dimensional. We have arrived to a contradiction with the equality $F_0 = \varphi$.

Case 2. The sequence $\{\deg JPu_k\}$ is unbounded. Since $N = (L + Z + D(Z)) \cap Y$ is finite dimensional, $m = \sup\{\deg Jw : w \in N \setminus \{0\}\}$ is finite: $m \in \mathbb{Z}_+$. We shall show that $\deg JPu_{k+1} = n + \deg JPu_k$ whenever $\deg JPu_k > m$. Indeed, let $k \in \mathbb{Z}_+$ be such that $\deg JPu_k > m$. By definition of P , $u_k - Pu_k \in Z$ and $u_{k+1} - Pu_{k+1} \in Z$. As we know, $u_{k+1} \in Du_k + L$. Hence $Pu_{k+1} \in DPu_k + L + Z + D(Z)$. Since Pu_{k+1} and DPu_k belong to Y , we have $Pu_{k+1} \in DPu_k + N$. Thus there is $w \in N$ such that $Pu_{k+1} = DPu_k + w$. Hence $JPu_{k+1} = JDPu_k + Jw = M^nJPu_k + Jw$. Since $\deg M^nJPu_k = n + \deg JPu_k > m \geq \deg Jw$, we have $\deg JPu_{k+1} = n + \deg JPu_k$. Since $\{\deg JPu_k\}$ is unbounded, there is $k \in \mathbb{Z}_+$ such that $\deg JPu_k > m$ and according to the just proven statement, we have $\deg JPu_j = \deg JPu_k + n(j - k)$ for $j \geq k$. Since any family of polynomials with pairwise different degrees is linearly independent, JPu_j for $j \geq k$ are linearly independent. Since JP is a linear operator, u_j for $j \geq k$ are linearly independent. Hence the sequence of spaces $\{F_j\}_{j \geq k}$ is strictly decreasing. On the other hand, we know that $F_j = \varphi$ for each $j \in \mathbb{Z}_+$. This contradiction completes the proof.

4 Hypercyclic operators on direct sums

Lemma 4.1. *Let X and Y be topological vector spaces such that there exists $T \in L(X, Y \times \mathbb{K})$ with dense range. Then for any closed hyperplane H of X , there exists $S \in L(H, Y)$ with dense range.*

Proof. We can express the restriction $T_0 \in L(H, Y \times \mathbb{K})$ of T to H as $T_0 = (S_0, g)$, where $S_0 \in L(H, Y)$ and $g \in H'$. If T_0 has dense range, then $S = S_0$ is a required operator. It remains to consider the case when the range of T_0 is not dense. Since the range of T is dense and T_0 is a restriction of T to a closed hyperplane, the codimension of $\overline{T_0(H)}$ in $Y \times \mathbb{K}$ does not exceed 1. Hence this codimension is exactly 1 and there is a non-zero $\psi \in (Y \times \mathbb{K})'$ such that $\overline{T_0(H)} = \ker \psi$. If $\ker \psi = Y \times \{0\}$, then again we can take $S = S_0$. If $\ker \psi \neq Y \times \{0\}$, there is $y \in Y$ such that $\psi(y) = 1$. It is straightforward to verify that $S \in L(H, Y)$, $Sx = S_0x + g(x)y$ has dense range. \square

Lemma 4.2. *Let $\{X_n\}_{n \in \mathbb{Z}_+}$ be a sequence of infinite dimensional locally convex spaces such that*

- (4.2.1) *there is a sequence $\{U_n\}_{n \in \mathbb{Z}_+}$ of subsets of X_0 such that $\text{span}(U_n)$ is dense in X_0 for each $n \in \mathbb{Z}_+$ and for any non-empty open subset U of X_0 , there is $m \in \mathbb{Z}_+$ for which $U_m \subseteq U$;*
- (4.2.2) *there exists $T_0 \in L(X_0, X_0 \oplus X_1)$ with dense range;*
- (4.2.3) *for each $n \in \mathbb{N}$, there exists $T_n \in L(X_n, X_{n+1} \times \mathbb{K})$ with dense range.*

Then there is a hypercyclic operator S on $X = \bigoplus_{n=0}^{\infty} X_n$.

Proof. Let $Z_n = \{x \in X : x_j = 0 \text{ for } j > n\}$ for $n \in \mathbb{Z}_+$. Clearly X is the union of the increasing sequence of subspaces Z_n and each Z_n is naturally isomorphic to the direct sum of X_k for $0 \leq k \leq n$. We shall construct inductively a sequence of operators $S_k \in L(Z_k, Z_{k+1})$ and vectors $y_k \in X_0$ satisfying the following conditions for any $k \in \mathbb{Z}_+$:

- (a1) $S_j = S_k|_{Z_j}$ for $0 \leq j < k$; (a3) $S_k \dots S_0 y_k \notin Z_k$; (a5) $y_k \in U_k$.
- (a2) $S_k(Z_k)$ is dense in Z_{k+1} ; (a4) $S_k \dots S_0 y_{k-1} = y_k$ if $k \geq 1$;

By (4.2.2), there is $S_0 \in L(Z_0, Z_1)$ with dense range. Since Z_0 is a proper closed subspace of Z_1 and $\text{span}(U_0)$ is dense in $X_0 = Z_0$, we can pick $y_0 \in U_0$ such that $S_0 y_0 \notin Z_0$. The basis of induction has been constructed. Assume that $n \in \mathbb{N}$ and $y_k \in X_0$, $S_k \in L(Z_k, Z_{k+1})$ satisfying (a1–a5) for $k < n$ are already constructed. By (a3) for $k = n - 1$, $w = S_{n-1} \dots S_0 y_{n-1} \notin Z_{n-1}$. That is, the n^{th} component w_n of w is non-zero. Since X_n is locally convex, we can pick a closed hyperplane H in X_n such that $w_n \notin H$. Let $P \in L(Z_n)$ be the linear projection onto H along $Z_{n-1} \oplus \text{span}\{w_n\}$. By (4.2.3) and Lemma 4.1, there is $R \in L(H, X_{n+1})$ with dense range. According to (a3) for $k = n - 1$, $S_{n-1} \dots S_0(Z_0)$ is dense in Z_n . Hence $Q(Z_0)$ is dense in X_{n+1} , where $Q = RPS_{n-1} \dots S_0$. Since $\text{span}(U_n)$ is dense in Z_0 , we can pick $y_n \in U_n$ such that $Qy_n \neq 0$. Since $Z_n = Z_{n-1} \oplus H \oplus \text{span}\{w\}$, we define the linear map $S_n : Z_n \rightarrow Z_{n+1}$ by the formula

$$S_n(x + y + sw) = S_{n-1}x + Ry + sy_n \text{ for } x \in Z_{n-1}, y \in H \text{ and } s \in \mathbb{K}.$$

The operator S_n is continuous since S_{n-1} and R are continuous. Clearly (a1) and (a5) for $k = n$ are satisfied. Next, $S_n(Z_n) \supseteq S_{n-1}(Z_{n-1}) + R(H)$. By (a2) for $k = n - 1$, $S_{n-1}(Z_{n-1})$ is dense in Z_n . Since $R(H)$ is dense in X_{n+1} , $S_n(Z_n)$ is dense in $Z_{n+1} = Z_n \oplus X_n$, which gives us (a2) for $k = n$. Since $Qy_n \neq 0$, the last display implies (a3) for $k = n$. Finally, since $S_n w = y_n$ from the definition of w we get (a4) for $k = n$. The inductive construction of S_k and y_k is complete.

Condition (a1) ensures that there is a unique $S \in L(X)$ such that $S|_{Z_n} = S_n$ for any $n \in \mathbb{Z}_+$. By (a4), $S^{k+1}y_{k-1} = y_k$ for each $k \in \mathbb{Z}_+$ and therefore $A = \{y_n : n \in \mathbb{Z}_+\}$ is contained in the orbit $O = \{S^n y_0 : n \in \mathbb{Z}_+\}$. By (a5) and (4.2.1), A is dense in $X_0 = Z_0$. By (a2), $S^m(A)$ is dense in Z_m for each $m \in \mathbb{Z}_+$. Since $A \subset O$, we have $S^m(A) \subset O$ and therefore $O \cap Z_m$ is dense in Z_m for any $m \in \mathbb{Z}_+$. Hence O is dense in X . That is, y_0 is a hypercyclic vector for S . \square

Remark 4.3. Condition (4.2.1) is satisfied if there exists a dense linear subspace Y of X_0 , carrying a topology, stronger than the one inherited from X_0 and turning Y into a separable metrizable topological vector space. Indeed any countable base $\{U_n\}_{n \in \mathbb{Z}_+}$ of topology of Y satisfies (4.2.1). In this case, the orbit O in the proof of Lemma 4.2 is not just dense. It is sequentially dense. The latter property is strictly stronger than density already for countable direct sums of separable infinite dimensional Banach spaces.

Remark 4.4. Lemma 4.2 remains true (with virtually the same proof) if we replace locally convex direct sum by the direct sum in the category of topological vector spaces. In the latter case the condition of local convexity of X_n can be replaced by the weaker condition that X'_n separates points of X_n for each $n \in \mathbb{Z}_+$.

4.1 Proof of Theorem 1.10

Let X be the direct sum of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of locally convex spaces such that the topology of each X_n is not weak and each X_n admits an ℓ_1 -sequence with dense linear span. By Lemma 1.2, there is a Banach disk D in X_0 such that $(X_0)_D$ is separable and is dense in X_0 . By Remark 4.3, (4.2.1) is satisfied. From Remark 2.4 and Lemma 2.3 it follows that (4.2.2) and (4.2.3) are also satisfied. By Lemma 4.2, there is a hypercyclic $T \in L(X)$.

4.2 Proof of Theorem 1.9

Lemma 4.5. *Let X and Y be separable infinite dimensional Fréchet spaces. Then there is no $T \in L(X, Y)$ with dense range if and only if X is isomorphic to ω and Y is not isomorphic to ω .*

Proof. If both X and Y are isomorphic to ω , then there is a surjective $T \in L(X, Y)$. If X is isomorphic to ω , Y is not and $T \in L(X, Y)$, then $Z = T(X)$ carries minimal locally convex topology [4] since ω does. It follows that Z is either finite dimensional or isomorphic to ω and therefore complete. Hence Z is closed in Y and $Z = \overline{Z} \neq Y$ since Y is neither finite dimensional nor isomorphic to ω . Thus there is no $T \in L(X, Y)$ with dense range. It remains to show that there is $T \in L(X, Y)$ with dense range

if X is not isomorphic to ω . In the latter case the topology of X is not weak and it remains to apply Lemma 2.3 since any separable Fréchet space admits an ℓ_1 -sequence with dense linear span. \square

Lemma 4.6. *Let X be the countable locally convex direct sum of separable Fréchet spaces infinitely many of which are infinite dimensional. Then X is isomorphic to the locally convex direct sum of a sequence $\{Y_n\}_{n \in \mathbb{Z}_+}$ of separable infinite dimensional Fréchet spaces such that either Y_n is isomorphic to ω for each $n \geq 1$ or Y_n is non-isomorphic to ω for each $n \in \mathbb{Z}_+$.*

Proof. We know that X is the direct sum of $\{X_\alpha\}_{\alpha \in A}$, where A is countable, each X_α is a separable Fréchet space and X_α is infinite dimensional for infinitely many $\alpha \in A$. If the set B of $\alpha \in A$ such that X_α is infinite dimensional and non-isomorphic to ω is infinite, we can write $A = \{\alpha_n : n \in \mathbb{Z}_+\}$, where α_n are pairwise different and $\alpha_n \in B$ for each $n \in \mathbb{Z}_+$. Then X is isomorphic to the direct sum of $Y_n = X_{\alpha_n} \oplus X_{\alpha_{-n-1}}$ for $n \in \mathbb{Z}_+$ and each Y_n is a separable infinite dimensional Fréchet space non-isomorphic to ω .

If B is finite, the set C of $\alpha \in A$ for which X_α is isomorphic to ω is infinite. Hence we can write $A \setminus B = \{\alpha_n : n \in \mathbb{Z}\}$, where α_n are pairwise different and $\alpha_n \in C$ for each $n \in \mathbb{Z}_+$. Let $Y_0 = X_{\alpha_0} \oplus \bigoplus_{\alpha \in B} X_\alpha$ and $Y_n = X_{\alpha_n} \oplus X_{\alpha_{-n}}$ for $n \in \mathbb{N}$. Since B is finite and X_{α_0} is isomorphic to ω , Y_0 is a separable infinite dimensional Fréchet space. Since for each $n \in \mathbb{N}$, X_{α_n} is isomorphic to ω and $X_{\alpha_{-n}}$ is either finite dimensional or isomorphic to ω , Y_n is isomorphic to ω for any $n \in \mathbb{N}$. It remains to notice that X is isomorphic to the direct sum of the sequence $\{Y_n\}_{n \in \mathbb{Z}_+}$. \square

We are ready to prove Theorem 1.9. Let X be a countable infinite direct sum of separable Fréchet spaces. If all the spaces in the sum, except for finitely many, are finite dimensional, then X is isomorphic to $Y \times \varphi$, where Y is a Fréchet space. By Theorem 1.8, X admits no cyclic operator with dense range. In particular, there are no supercyclic operators on X . If there are infinitely many infinite dimensional spaces in the sum defining X , then according to Lemma 4.6, X is isomorphic to the locally convex direct sum of a sequence $\{Y_n\}_{n \in \mathbb{Z}_+}$ of separable infinite dimensional Fréchet spaces such that either all Y_n are non-isomorphic to ω or all Y_n for $n \geq 1$ are isomorphic to ω . In any case, by Lemma 4.5, there exists $T_0 \in L(Y_0, Y_0 \oplus Y_1)$ with dense range and there exist $T_n \in L(Y_n, Y_{n+1} \times \mathbb{K})$ with dense ranges for all $n \in \mathbb{N}$. By Lemma 4.2 and Remark 4.3, there is a hypercyclic operator on X . The proof of Theorem 1.9 is complete.

5 Hypercyclic operators on countable unions of spaces

Lemma 5.1. *Let a locally convex space X be the union of an increasing sequence $\{X_n\}_{n \in \mathbb{N}}$ of its closed linear subspaces. Assume also that for any $n \in \mathbb{N}$ there is an ℓ_1 -sequence with dense span in X_n and the topology of X_n/X_{n-1} is not weak, where $X_0 = \{0\}$. Then there exists a linear map $S : X \rightarrow X$ and $x_0 \in X_1$ such that $S|_{X_n} \in L(X_n, X_{n+1})$ for any $n \in \mathbb{N}$ and $\{S^k x_0 : k \in \mathbb{Z}_+\}$ is dense in X .*

Note that we do not claim continuity of S on X . Although if, for instance, X is the inductive limit of the sequence $\{X_n\}$, then continuity of S immediately follows from the continuity of the restrictions $S|_{X_n}$.

Proof of Lemma 5.1. For $n \in \mathbb{N}$, let $\{x_{n,k}\}_{k \in \mathbb{Z}_+}$ be an ℓ_1 -sequence with dense linear span in X_n . For any $n \in \mathbb{N}$, we apply Lemma 2.1 with $(Y, Y_1, Y_0) = (X, X_n, X_{n-1})$ to obtain a uniformly equicontinuous sequence $\{f_{n,k}\}_{k \in \mathbb{Z}_+}$ in X' such that each $f_{n,k}$ vanishes on X_{n-1} and $\varphi \subseteq \{\{f_{n,k}(x)\}_{k \in \mathbb{Z}_+} : x \in X_n\}$. By Lemma 1.2, there is a Banach disk D in X such that X_D is a dense subspace of X_1 and the Banach space X_D is separable. Let $\{U_n\}_{n \in \mathbb{N}}$ be a base of topology of X_D . We shall construct inductively a sequence of operators $S_k \in L(X, X_{k+1})$ and vectors $y_k \in X_D$ satisfying the following conditions for any $k \in \mathbb{N}$:

- (b1) $S_j|_{X_j} = S_k|_{X_j}$ for $1 \leq j < k$; (b3) $f_{k+1,0}(S_k \dots S_1 y_k) \neq 0$; (b5) $y_k \in U_k$.
- (b4) $S_k \dots S_1 y_{k-1} = y_k$ if $k \geq 2$; (b2) $S_k(X_k)$ is dense in X_{k+1} ;

Consider the linear map $S_1 : X \rightarrow X_2$ defined by the formula

$$S_1 x = \sum_{k=0}^{\infty} 2^{-k} f_{1,k}(x) x_{2,k}.$$

Since $\{x_{2,k}\}_{k \in \mathbb{Z}_+}$ is an ℓ_1 -sequence in X_2 and $\{f_{1,k} : k \in \mathbb{Z}_+\}$ is uniformly equicontinuous, the above display defines a continuous linear operator from X to X_2 . Since $\varphi \subseteq \{\{f_{1,k}(x)\}_{k \in \mathbb{Z}_+} : x \in X_1\}$, $S_1(X_1)$ contains $\text{span}\{x_{2,k} : k \in \mathbb{Z}_+\}$. Hence $S_1(X_1)$ is dense in X_2 . Since X_D is dense in X_1 , S_1 has dense range and $X_2 \cap \ker f_{2,0}$ is nowhere dense in X_2 , we can pick $y_1 \in U_1$ such that $f_{2,0}(S_1 y_1) \neq 0$. The basis of induction has been constructed. Assume now that $n \geq 2$ and $y_k \in X_D$, $S_k \in L(X, X_{k+1})$, satisfying (b1–b5) for $k \leq n-1$, are already constructed. According to (b3) for $k = n-1$, $f_{n,0}(w) \neq 0$, where $w = R y_{n-1}$ and $R = S_{n-1} \dots S_1$. Since $H = X_n \cap \ker f_{n,0}$ is a closed hyperplane in X_n and $w \notin H$, we have $X_n = H \oplus \text{span}\{w\}$. Let $H_0 = H \cap \ker f_{n,1}$. Then H_0 is a closed hyperplane of H . By (b2) for $k = n-1$, $R(X_1)$ is dense in X_n . Since X_D is dense in X_1 , we can pick $y_n \in U_n$ such that $u = R y_n \notin H_0 \oplus \text{span}\{w\}$. Thus $X_n = H_0 \oplus \text{span}\{u, w\}$. Pick any $v \in X_{n+1}$ such that $f_{n+1,0}(v) \neq 0$ and let

$$x_0 = y_n - S_{n-1} w - \sum_{k=0}^{\infty} 2^{-k} f_{n,k+2}(w) x_{n+1,k} \quad \text{and} \quad x_1 = v - S_{n-1} u - \sum_{k=0}^{\infty} 2^{-k} f_{n,k+2}(u) x_{n+1,k}.$$

The above series converge since $\{x_{n+1,k}\}_{k \in \mathbb{Z}_+}$ is an ℓ_1 -sequence and $\{f_{n,k} : k \in \mathbb{Z}_+\}$ is uniformly equicontinuous. By construction of H_0 , u and w , the matrix $\begin{pmatrix} f_{n,0}(w) & f_{n,1}(w) \\ f_{n,0}(u) & f_{n,1}(u) \end{pmatrix}$ is invertible. This allows us to find $y_0, y_1 \in \text{span}\{x_0, x_1\} \subset X_{n+1}$ satisfying

$$f_{n,0}(w) y_0 + f_{n,1}(w) y_1 = x_0 \quad \text{and} \quad f_{n,0}(u) y_0 + f_{n,1}(u) y_1 = x_1.$$

Consider the linear map $S_n : X \rightarrow X_{n+1}$ defined by the formula

$$S_n x = S_{n-1} x + f_{n,0}(x) y_0 + f_{n,1}(x) y_1 + \sum_{k=0}^{\infty} 2^{-k} f_{n,k+2}(x) x_{n+1,k}.$$

The above display defines a continuous linear operator since $\{x_{n+1,k}\}_{k \in \mathbb{Z}_+}$ is an ℓ_1 -sequence and $\{f_{n,k} : k \in \mathbb{Z}_+\}$ is uniformly equicontinuous. By the last three displays, $S_n w = y_n$ and $S_n u = v$. From definition of w and u and the relation $f_{n+1,0}(v) \neq 0$ it follows that (b3) and (b4) for $k = n$ are satisfied. Clearly (b5) for $k = n$ is also satisfied. Since each $f_{n,k}$ vanishes on X_{n-1} , we have from the last display that $S_n x = S_{n-1} x$ for any $x \in X_{n-1}$. Hence (b1) for $k = n$ is satisfied. It remains to verify (b2) for $k = n$. Let U be a non-empty open subset of X_{n+1} . Since $E = \text{span}\{x_{n+1,k} : k \in \mathbb{Z}_+\}$ is dense in X_{n+1} , we can find $x \in E$ and a convex balanced neighborhood W of zero in X_{n+1} such that $x + W \subseteq U$. Since $\varphi \subseteq \{\{f_{n,k}(x)\}_{k \in \mathbb{Z}_+} : x \in X_n\}$ and $x \in E = \text{span}\{x_{n+1,k} : k \in \mathbb{Z}_+\}$, we can pick $y \in X_n$ such that $f_{n,0}(y) = f_{n,1}(y) = 0$ and $x = \sum_{k=0}^{\infty} 2^{-k} f_{n,k+2}(y) x_{n+1,k}$. Hence $S_n y = S_{n-1} y + x$. By (b2) for $k = n-1$, $S_{n-1}(X_{n-1})$ is dense in X_n . Since $S_{n-1} y \in X_n$, we can find $r \in X_{n-1}$ such that $S_{n-1} r \in S_{n-1} y - W$. By the already proven property (b1) for $k = n$, $S_{n-1} r = S_n r$. Hence $S_n r \in S_{n-1} y - W$. Using the equality $S_n y = S_{n-1} y + x$, we get $S_n(y - r) \in x + W \subseteq U$. Hence any non-empty open subset of X_{n+1} contains elements of $S_n(X_n)$, which proves (b2) for $k = n$. The inductive construction of S_k and y_k is complete.

By (b2), there is a unique linear map $S : X \rightarrow X$ such that $S|_{X_n} = S_n|_{X_n}$ for any $n \in \mathbb{N}$. By (b4), $S^{k+1} y_k = y_{k+1}$ for each $k \in \mathbb{N}$. Hence $A = \{y_n : n \in \mathbb{N}\}$ is contained in $O = \{S^n y_1 : n \in \mathbb{Z}_+\}$. By (b5), A is dense in X_D and therefore is dense in X_1 . By (b2), $S^m(A)$ is dense in X_{m+1} for each $m \in \mathbb{Z}_+$. Since $A \subset O$, we have $S^m(A) \subset O$ and therefore $O \cap X_m$ is dense in X_m for each $m \in \mathbb{N}$. Hence O is dense in X . Thus the required condition is satisfied with $x_0 = y_1$. \square

Before proving Theorem 1.7, we need to make the following two elementary observations.

Lemma 5.2. *Let X be an LB-space and Y be a closed linear subspace of X . Then either X/Y is finite dimensional or the topology of X/Y is not weak.*

Proof. Since X is an LB-space, it is the inductive limit of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of Banach spaces. If X/Y is infinite dimensional, we can find a linearly independent sequence $\{f_n\}_{n \in \mathbb{Z}_+}$ in X' such that each f_n vanishes on Y . Next, we pick a sequence $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ of positive numbers converging to zero fast enough to ensure that $\varepsilon_n \|f_n|_{X_k}\|_k \rightarrow 0$ as $n \rightarrow \infty$ for each $k \in \mathbb{N}$. It follows that $\varepsilon_n f_n$ pointwise converge to zero on X . Since any LB-space is barrelled [12, 4], $\{\varepsilon_n f_n : n \in \mathbb{Z}_+\}$ is uniformly equicontinuous. Hence $p(x) = \sup\{\varepsilon_n |f_n(x)| : n \in \mathbb{Z}_+\}$ is a continuous seminorm on X . Since each f_n vanishes on Y , $Y \subseteq \ker p$. Then $\tilde{p}(x + Y) = p(x)$ is a continuous seminorm on X/Y . Since f_n are linearly independent, $\ker p$ has infinite codimension in X and therefore $\ker \tilde{p}$ has infinite codimension in X/Y . Hence the topology of X/Y is not weak. \square

Lemma 5.3. *Let X be an inductive limit of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of Banach spaces such that X_0 is dense in X . Then X has no quotients isomorphic to φ .*

Proof. Assume that X has a quotient isomorphic to φ . By Lemma 3.2, X is isomorphic to $Y \times \varphi$ for some closed linear subspace Y of X . Let $J : X_0 \rightarrow X$ be the natural embedding. Since X_0 is dense in X , J has dense range. Hence $J' : X' \rightarrow X'_0$ is injective. Since X is isomorphic to $Y \times \varphi$, X' is isomorphic to $Y' \times \omega$ (ω is naturally isomorphic to φ' ; here X' , X'_0 , Y' and φ' carry strong topology [12, 4]). Hence, there exists an injective continuous linear operator from ω to the Banach space X'_0 . That is impossible, since any injective continuous linear operator from ω to a locally convex space is an isomorphism onto image and ω is non-normable. \square

5.1 Proof of Theorem 1.7

Throughout this section X is the inductive limit of a sequence $\{X_n\}_{n \in \mathbb{Z}_+}$ of separable Banach spaces. Let also \overline{X}_n be the closure of X_n in X . First, we shall prove that (1.7.4) implies (1.7.3). Assume that (1.7.4) is satisfied. Then we can pick a strictly increasing sequence $\{n_k\}_{k \in \mathbb{Z}_+}$ of non-negative integers such that $0 < \dim \overline{X}_{n_{k+1}} / \overline{X}_{n_k} < \infty$ for each $k \in \mathbb{Z}_+$. Hence, for any $k \in \mathbb{Z}_+$, there is a non-trivial finite dimensional subspace Y_k of $X_{n_{k+1}}$ such that $\overline{X}_{n_k} \oplus Y_k = \overline{X}_{n_{k+1}}$. Thus the vector space X can be written as an algebraic direct sum

$$X = \overline{X}_{n_0} \oplus \bigoplus_{k=0}^{\infty} Y_k = \bigcup_{k=0}^{\infty} \overline{X}_{n_0} \oplus Z_k, \quad \text{where} \quad Z_k = \bigoplus_{j=0}^{k-1} Y_j. \quad (5.1)$$

Apart from the original topology τ on X , we can consider the topology θ , turning the sum (5.1) into a locally convex direct sum. Obviously $\tau \subseteq \theta$. On the other hand, if W is a balanced convex θ -neighborhood of 0 in X , then $W \cap \overline{X}_{n_k}$ is a τ -neighborhood of zero in \overline{X}_{n_k} for any $k \in \mathbb{Z}_+$. Indeed, it follows from the fact that $\overline{X}_{n_k} = \overline{X}_{n_0} \oplus Z_k$, where Z_k is finite dimensional. Since the topology of each X_{n_k} is stronger than the one inherited from X , $W \cap X_{n_k}$ is a neighborhood of zero in X_{n_k} for each $k \in \mathbb{Z}_+$. Since X is the inductive limit of the sequence $\{X_{n_k}\}_{k \in \mathbb{Z}_+}$, W is a τ -neighborhood of zero in X . Hence $\theta \subseteq \tau$. Thus $\theta = \tau$ and therefore X is isomorphic to $\overline{X}_{n_0} \times Y$, where Y is the locally convex direct sum of Y_k for $k \in \mathbb{Z}_+$. Since Y_k are finite dimensional, Y is isomorphic to φ . Since \overline{X}_{n_0} is the inductive limit of the sequence $\{\overline{X}_{n_0} \cap X_{n_k}\}_{k \in \mathbb{Z}_+}$ of separable Banach spaces (with the topology inherited from X_{n_k}), the first one of which is dense, (1.7.3) is satisfied. Thus (1.7.4) implies (1.7.3).

Assume now that (1.7.3) is satisfied. By Lemma 5.3, Y has no quotients isomorphic to φ . By Theorem 1.8, there are no cyclic operators with dense range on X . Thus (1.7.3) implies (1.7.2). The implication (1.7.2) \implies (1.7.1) is obvious since any hypercyclic operator is cyclic and has dense range. It remains to show that (1.7.1) implies (1.7.4). Assume the contrary. That is, (1.7.1) is satisfied and (1.7.4) fails. The latter means that either there is $n \in \mathbb{Z}_+$ such that \overline{X}_n is dense in X or there is a strictly increasing sequence $\{n_k\}_{k \in \mathbb{Z}_+}$ of non-negative integers such that $\overline{X}_{n_{k+1}} / \overline{X}_{n_k}$ is infinite dimensional for each $k \in \mathbb{Z}_+$. In the first case, X supports an ℓ_1 -sequence with dense linear span.

By Lemma 5.2, the topology of X is not weak. By Theorem 1.6, there is a hypercyclic operator on X , which contradicts (1.7.1). It remains to consider the case when there exists a strictly increasing sequence $\{n_k\}_{k \in \mathbb{Z}_+}$ of non-negative integers such that $\overline{X}_{n_{k+1}}/\overline{X}_{n_k}$ is infinite dimensional for each $k \in \mathbb{Z}_+$. By Lemma 5.2, the topology of each $\overline{X}_{n_{k+1}}/\overline{X}_{n_k}$ is not weak. Since each X_{n_k} is a separable Banach space, there is an ℓ_1 -sequence $\{x_{k,j}\}_{j \in \mathbb{Z}_+}$ in X_{n_k} with dense span. The same sequence is an ℓ_1 -sequence with dense span in \overline{X}_{n_k} . By Lemma 5.1, there is a linear map $S : X \rightarrow X$ and $x_0 \in X$ such that $\{S^k x_0 : k \in \mathbb{Z}_+\}$ is dense in X and the restriction of S to each \overline{X}_{n_k} is continuous. Since the topology of X_{n_k} is stronger than the one inherited from X , the restriction of S to each X_{n_k} is a continuous linear operator from X_{n_k} to X . Since X is the inductive limit of $\{X_{n_k}\}_{k \in \mathbb{Z}_+}$, $S : X \rightarrow X$ is continuous. Hence S is a hypercyclic continuous linear operator on X , which contradicts (1.7.1). The proof of the implication (1.7.1) \implies (1.7.4) and that of Theorem 1.7 is complete.

6 Remarks on mixing versus hereditarily hypercyclic

We start with the following remark. As we have already mentioned, φ supports no supercyclic operators [5], which follows also from Theorem 1.8. On the other hand, φ supports a transitive operator [6]. The latter statement can be easily strengthened. Namely, take the backward shift T on φ . That is, $Te_0 = 0$ and $Te_n = e_{n-1}$ for $n \geq 1$, where $\{e_n\}_{n \in \mathbb{Z}_+}$ is the standard basis in φ . By Theorem 2.2 from [2], $I + T$ is mixing. Thus we have the following proposition.

Proposition 6.1. *φ supports a mixing operator and supports no supercyclic operators.*

On the other hand, a topological vector space of countable algebraic dimension can support a hypercyclic operator, as observed by several authors, see [6], for instance. The following proposition formalizes and extends this observation.

Proposition 6.2. *Let X be a normed space of countable algebraic dimension. Then there exists a hypercyclic mixing operator $T \in L(X)$.*

Proof. The completion \overline{X} of X is a separable infinite dimensional Banach space. By Theorem 1.6, there is a hereditarily hypercyclic operator $S \in L(\overline{X})$. Let $x \in \overline{X}$ be a hypercyclic vector for S and $E = \text{span}\{S^n x : n \in \mathbb{Z}_+\}$. Grivaux [8] demonstrated that for any two countably dimensional dense linear subspaces E_1 and E_2 of a separable infinite dimensional Banach space Y , there is an isomorphism $J : Y \rightarrow Y$ such that $J(E_1) = E_2$. Hence there is an isomorphism $J : \overline{X} \rightarrow \overline{X}$ such that $J(X) = E$. Let $T_0 = J^{-1}SJ$. Since $J(X) = E$ and E is S -invariant, X is T_0 -invariant. Thus the restriction T of T_0 to X is a continuous linear operator on X . Moreover, since the S -orbit of x is dense in \overline{X} , the T_0 -orbit of $J^{-1}x$ is dense in \overline{X} . Since $J^{-1}x \in X$, the latter orbit is exactly the T -orbit of $J^{-1}x$ and $J^{-1}x$ is hypercyclic for T . Hence T is hypercyclic. Next, T_0 is mixing since it is similar to the mixing operator S . Hence T is mixing as a restriction of a mixing operator to a dense invariant subspace. \square

By Proposition 1.3, if a topological vector space X is Baire separable and metrizable, then any mixing $T \in L(X)$ is hereditarily hypercyclic. By Proposition 6.2, there are mixing operators on any countably dimensional normed space. The next theorem implies that there are no hereditarily hypercyclic operators on countably dimensional topological vector spaces, emphasizing the necessity of the Baire condition in Proposition 1.3.

Theorem 6.3. *Let X be a topological vector space such that there exists a hereditarily universal family $\{T_n : n \in \mathbb{Z}_+\} \subset L(X)$. Then $\dim X > \aleph_0$.*

Proof. Since the topology of any topological vector space can be defined by a family of quasinorms [12], we can pick a non-zero continuous quasinorm p on X . That is, $p : X \rightarrow [0, \infty)$ is non-zero, continuous, $p(0) = 0$, $p(x+y) \leq p(x) + p(y)$ and $p(zx) \leq p(x)$ for any $x, y \in X$ and $z \in \mathbb{K}$ with $|z| \leq 1$ and (X, τ_p) is a (not necessarily Hausdorff) topological vector space, where τ_p is the topology defined

by the pseudometric $d(x, y) = p(x - y)$. The latter property implies that $p(tx_n) \rightarrow 0$ for any $t \in \mathbb{K}$ and any sequence $\{x_n\}_{n \in \mathbb{Z}_+}$ in X such that $p(x_n) \rightarrow 0$.

Let \varkappa be the first uncountable ordinal (usually denoted ω_1). We construct sequences $\{x_\alpha\}_{\alpha < \varkappa}$ and $\{A_\alpha\}_{\alpha < \varkappa}$ of vectors in X and subsets of \mathbb{Z}_+ respectively such that for any $\alpha < \varkappa$,

- (s1) A_α is infinite and x_α is a universal vector for $\{T_n : n \in A_\alpha\}$;
- (s2) $p(T_n x_\beta) \rightarrow 0$ as $n \rightarrow \infty$, $n \in A_\alpha$ for any $\beta < \alpha$;
- (s3) $A_\alpha \setminus A_\beta$ is finite for any $\beta < \alpha$.

For the basis of induction we take $A_0 = \mathbb{Z}_+$ and x_0 being a universal vector for $\{T_n : n \in \mathbb{Z}_+\}$. It remains to describe the induction step. Assume that $\gamma < \varkappa$ and x_α, A_α satisfying (s1–s3) for $\alpha < \gamma$ are already constructed. We have to construct x_γ and A_γ satisfying (s1–s3) for $\alpha = \gamma$.

Case 1: γ has the immediate predecessor. That is $\gamma = \rho + 1$ for some ordinal $\rho < \varkappa$. Since x_ρ is universal for $\{T_n : n \in A_\rho\}$, we can pick an infinite subset $A_\gamma \subset A_\rho$ such that $p(T_n x_\rho) \rightarrow 0$ as $n \rightarrow \infty$, $n \in A_\gamma$. Since A_γ is contained in A_ρ , from (s3) for $\alpha \leq \rho$ it follows that $A_\gamma \setminus A_\beta$ is finite for any $\beta < \gamma$. Hence (s3) for $\alpha = \gamma$ is satisfied. Now from (s3) for $\alpha = \gamma$ and (s2) for $\alpha < \gamma$ it follows that (s2) is satisfied for $\alpha = \gamma$. Finally, since $\{T_n : n \in \mathbb{Z}_+\}$ is hereditarily universal, we can pick $x_\gamma \in X$ universal for $\{T_n : n \in A_\gamma\}$. Hence (s1) for $\alpha = \gamma$ is also satisfied.

Case 2: γ is a limit ordinal. Since γ is a countable ordinal, there is a strictly increasing sequence $\{\alpha_n\}_{n \in \mathbb{Z}_+}$ of ordinals such that $\gamma = \sup\{\alpha_n : n \in \mathbb{Z}_+\}$. Now pick consecutively n_0 from A_{α_0} , $n_1 > n_0$ from $A_{\alpha_0} \cap A_{\alpha_1}$, $n_2 > n_1$ from $A_{\alpha_0} \cap A_{\alpha_1} \cap A_{\alpha_2}$ etc. The choice is possible since by (s3) for $\alpha < \gamma$, each $A_{\alpha_0} \cap \dots \cap A_{\alpha_n}$ is infinite. Now let $A_\gamma = \{n_j : j \in \mathbb{Z}_+\}$. Since $A_\gamma \setminus A_{\alpha_j} \subseteq \{n_0, \dots, n_{j-1}\}$, $A_\gamma \setminus A_{\alpha_j}$ is finite for each $j \in \mathbb{Z}_+$. Now if $\beta < \gamma$, we can pick $j \in \mathbb{Z}_+$ such that $\beta < \alpha_j < \gamma$. Then $A_\gamma \setminus A_\beta \subseteq (A_\gamma \setminus A_{\alpha_j}) \cup (A_{\alpha_j} \setminus A_\beta)$ is finite by (s3) with $\alpha = \alpha_j$. Moreover, since A_γ is contained in A_{α_j} up to a finite set, from (s2) with $\alpha = \alpha_j$ it follows that $p(T_n x_\beta) \rightarrow 0$ as $n \rightarrow \infty$, $n \in A_\gamma$. Hence (s2) and (s3) for $\alpha = \gamma$ are satisfied. Finally, since $\{T_n : n \in \mathbb{Z}_+\}$ is hereditarily universal, we can pick $x_\gamma \in X$ universal for $\{T_n : n \in A_\gamma\}$. Hence (s1) for $\alpha = \gamma$ is also satisfied. This concludes the construction of x_α and A_α satisfying (s1–s3) for $\alpha < \varkappa$.

In order to prove that $\dim X > \aleph_0$, it suffices to show that vectors $\{x_\alpha\}_{\alpha < \varkappa}$ are linearly independent. Assume the contrary. Then there are $n \in \mathbb{N}$, $z_1, \dots, z_n \in \mathbb{K} \setminus \{0\}$ and ordinals $\alpha_1 < \dots < \alpha_n < \varkappa$ such that $z_1 x_{\alpha_1} + \dots + z_n x_{\alpha_n} = 0$. By (s2) with $\alpha = \alpha_n$, $p(T_k x_{\alpha_j}) \rightarrow 0$ as $k \rightarrow \infty$, $k \in A_{\alpha_n}$ for $1 \leq j < n$. Denoting $c_j = -z_j/z_n$ and using linearity of T_k , we obtain $T_k x_{\alpha_n} = c_1 T_k x_{\alpha_1} + \dots + c_{n-1} T_k x_{\alpha_{n-1}}$ for any $k \in \mathbb{Z}_+$. Since p is a quasinorm, we have

$$p(T_k x_{\alpha_n}) \leq \sum_{1 \leq j < n} p(c_j T_k x_{\alpha_j}) \rightarrow 0 \text{ as } k \rightarrow \infty, k \in A_{\alpha_n},$$

which contradicts universality of x_{α_n} for $\{T_k : k \in A_{\alpha_n}\}$ (= (s1) with $\alpha = \alpha_n$). \square

Corollary 6.4. *A topological vector space of countable algebraic dimension supports no hereditarily hypercyclic operators.*

It is worth noting that there are infinite dimensional separable normed spaces, which support no multicyclic or transitive operators. We call a continuous linear operator T on a topological vector space X *simple* if T has shape $T = zI + S$, where $z \in \mathbb{K}$ and S has finite rank. It is easy to see that a simple operator on an infinite dimensional topological vector space is never transitive or multicyclic. We say that a topological vector space X is *simple* if it is infinite dimensional and any $T \in L(X)$ is simple. Thus simple topological vector spaces support no multicyclic or transitive operators. Examples of simple separable infinite dimensional normed spaces can be found in [16, 11, 17, 10, 13, 14]. Moreover, according to Valdivia [16], in any separable infinite dimensional Fréchet space there is a dense simple hyperplane. All examples of this type existing in the literature with one exception [13] are constructed with the help of the axiom of choice and the spaces produced are not Borel measurable in their completions. In [13] there is a constructive example of a simple separable infinite dimensional pre-Hilbert space H which is a countable union of compact sets. We end this section by observing that the following question remains open.

Question 6.5. *Is there a hereditarily hypercyclic operator on a countable direct sum of separable infinite dimensional Banach spaces?*

The most of the above results rely upon the underlying space being locally convex or at least having plenty of continuous linear functionals and for a good reason. Recall that an infinite dimensional topological vector space X is called *rigid* if $L(X)$ consists only of the operators of the form λI for $\lambda \in \mathbb{K}$. Of course, $X' = \{0\}$ if X is rigid. Clearly there are no transitive or multicyclic operators on a rigid topological vector space. Since there exist rigid separable \mathcal{F} -spaces [9], there are separable infinite dimensional \mathcal{F} -spaces supporting no multicyclic or transitive operators. On the other hand, the absence of non-zero continuous linear functionals on a topological vector space does not guarantee the absence of hypercyclic operators on it. It is well-known [9] that the spaces $L_p[0, 1]$ for $0 \leq p < 1$ are separable \mathcal{F} -spaces with no non-zero continuous linear functionals. Consider $T \in L(L_p[0, 1])$, $Tf(x) = f(x/2)$. By [2, Theorem 2.2], $I + T$ is mixing and therefore hereditarily hypercyclic. If the supply of continuous linear functionals on a separable \mathcal{F} -space is large enough, the existence of hypercyclic operators is guaranteed. Indeed, in [15] it is shown that there are hereditary hypercyclic operators on any separable \mathcal{F} -space X with uncountable algebraic dimension of X' . The case of finite positive dimension of X' provides no challenge. Indeed, if X is a topological vector space with $0 < n = \dim X' < \infty$, then $L = \{x \in X : f(x) = 0 \text{ for each } f \in X'\}$ is a closed linear subspace of X of codimension n invariant for each $T \in L(X)$. According to Wengenroth [18] a hypercyclic operator has no closed invariant subspaces of finite positive codimension, while a supercyclic operator has no closed invariant subspaces of codimension $n \in \mathbb{N}$ with $n > 1$ if $\mathbb{K} = \mathbb{C}$ or $n > 2$ if $\mathbb{K} = \mathbb{R}$. Thus the following proposition holds.

Proposition 6.6. *A topological vector space X with $0 < \dim X' < \infty$ supports no hypercyclic operators. If additionally $\dim_{\mathbb{R}} X' > 2$, then there are no supercyclic operators on X .*

In particular, let $X = L_p[0, 1] \times \mathbb{K}^n$ for $0 \leq p < 1$ and $n \in \mathbb{N}$. Then X supports no hypercyclic operators and X supports no supercyclic operators if $\dim_{\mathbb{R}} \mathbb{K}^n > 2$. The following problem remains unsolved.

Question 6.7. *Characterize \mathcal{F} -spaces supporting hypercyclic operators.*

The same question can be asked about supercyclic or hereditarily hypercyclic operators. We would like to emphasize the following related question.

Question 6.8. *Assume that an \mathcal{F} -space X supports a hypercyclic operator. Is it true that X supports a hereditarily hypercyclic operator?*

Note that an \mathcal{F} -space X with countably dimensional dual can support a hereditarily hypercyclic operator. For instance, one can take $X = \omega$ or $X = L_p[0, 1] \times \omega$ with $0 \leq p < 1$. For all we have seen so far, we could have come up with the conjecture that any separable \mathcal{F} -space with infinite dimensional dual supports a hypercyclic operator. This conjecture turns out to be false.

Theorem 6.9. *There exists a separable \mathcal{F} -space X such that X' is infinite dimensional and X supports no multicyclic operators.*

We prove Theorem 6.9 with the help of the following observation.

Proposition 6.10. *Let X be a rigid topological vector space and Y be a topological vector space such that any operator in $L(Y, X)$ has finite rank and $L(X, Y) = \{0\}$. Then $X \times Y$ supports no multicyclic operators.*

Proof. Let $T \in L(X \times Y)$. Using the assumptions on the spaces X and Y , we see that T acts according to the formula $T(x, y) = (\lambda x + By, Cy)$, where $\lambda \in \mathbb{K}$, $C \in L(Y)$, $B \in L(Y, X)$ and $L = B(Y)$ is a finite dimensional subspace of X . It is easy to see that $T^n(x, y) = (A_n(x, y), C^n y)$ for each $n \in \mathbb{N}$, where $A_n \in L(X \times Y, X)$, $A_n(x, y) = \lambda^n x + B(C^{n-1} + \lambda C^{n-2} + \dots + \lambda^{n-1} I)y$. Clearly $A_n(x, y) \in \text{span}(L \cup \{x\})$ for any $x, y \in X \times Y$. It follows that for any subspace M of X , $A_n(M \times Y) \subseteq M + L$ for each $n \in \mathbb{N}$. Since $M + L$ is finite dimensional if M is finite dimensional, we immediately see that T is not multicyclic. \square

Corollary 6.11. *Let X be a rigid topological vector space with no subspaces isomorphic to ω . Then $X \times \omega$ supports no multicyclic operators.*

Proof. Since X is rigid, $X' = \{0\}$. Then $g \circ S = 0$ for any $S \in L(X, \omega)$ and $g \in \omega'$. Since ω' separates points of ω , $S = 0$. Hence $L(X, \omega) = \{0\}$. Next, let $T \in L(\omega, X)$ be of infinite rank. Then $\omega/\ker T$ is infinite dimensional. Since any infinite dimensional quotient of ω is isomorphic to ω , we factorizing out the space $\ker T$, arrive to an injective $T_0 \in L(\omega, X)$. Since ω is a minimal topological vector space [4], T_0 is an isomorphism between ω and $T_0(\omega) = T(\omega) \subseteq X$. That is, X has a subspace isomorphic to ω , which is a contradiction. Thus any $T \in L(\omega, X)$ has finite rank. It remains to apply Proposition 6.10 to complete the proof. \square

Proof of Theorem 6.9. In [9], one can find examples of separable rigid \mathcal{F} -spaces Z with no subspaces isomorphic to ω . For instance, there are rigid subspaces in $L_p[0, 1]$ for $0 < p < 1$. By Corollary 6.11, $X = Z \times \omega$ supports no multicyclic operators. On the other hand, X' is infinite dimensional. \square

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